311. **Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.**

Find, by Cardan's Method, the real root (4) of $x^3-6x^2+10x=8$.

Solution by G. I. HOPKINS, Instructor in Mathematics and Astronomy, High School, Manchester, N. H.

Substitute $y+2$ for x. Then $y^3-2y-4=0$.
Substitute $v+z$ for y. Then, from the well known formula,

$$v=\sqrt[3]{[2+\tfrac{10}{9}\sqrt{3}]} \text{ and } z=\sqrt[3]{[2-\tfrac{10}{9}\sqrt{3}]}.$$

$\therefore 2+\frac{10}{9}\sqrt{3}$ must be the cube of a binomial, the first term of which is 1, and the second term contains $\sqrt{3}$.

Assume $(1+a\sqrt{3})^3=2+\frac{10}{9}\sqrt{3}$; whence $1+3a\sqrt{3}+9a^2+3a^3\sqrt{3}=2+\frac{10}{9}\sqrt{3}$, or $3a(1+a^2)\sqrt{3}+9a^2=1+\frac{10}{9}\sqrt{3}$.

$\therefore 9a^2=1$; whence $a=\frac{1}{3}$.

$\therefore 1+\frac{1}{3}\sqrt{3}$ is the cube root of $(2+\frac{10}{9}\sqrt{3})$.

$\therefore v=1+\frac{1}{3}\sqrt{3}$ and $z=1-\frac{1}{3}\sqrt{3}$; $v+z=2=y$; and, therefore, $x=y+2=4$.

Also by quadratics as follows:

Multiplying by x, $x^4-6x^3+10x^2-8x=0$.

$$x^4-6x^3+9x^2+x^2-8x=0,$$
$$(x^2-3x)^2+x^2-3x-5x=0,$$
$$(x^2-3x)^2+(x^2-3x)=5x.$$

Adding x^2-3x, $(x^2-3x)^2+2(x^2-3x)=x^2+2x$.

Adding 1, $(x^2-3x)^2+2(x^2-3x)+1=x^2+2x+1$.

$\therefore x^2-3x+1=\pm(x+1)$ and $x=4$.

Also solved by J. Scheffer, G. B. M. Zerr, G. W. Hartwell, and a student in Olivet College.

CALCULUS.

268. **Proposed by PROFESSOR R. D. CARMICHAEL, Anniston, Ala.**

Determine $\phi(y)$, independent of u, so that the equation $\int_0^u (u-y)^{(2p-1)/2}\phi(y)\,dy=u^m$ is satisfied, p and m being positive integers and $m>p$. Do you notice properties of special interest for any special cases?

Solution by the PROPOSER.

The problem may be generalized with small increase in the difficulty of solution; thus:

It is required to find $\phi(y)$ a function of y alone from the equation

$$(1)\qquad f(u)=\int_0^u h(u,\ y)u^a\,\phi(y)\,dy,$$

where $h(u, y)$ is a homogeneous function of u, y of degree zero such that it may be expressed as a finite or an infinite series,

$$h(u, y)=\beta_0+\beta_1\frac{y}{u}+\beta_2\frac{y^2}{u^2}+\beta_3\frac{y^3}{u^3}+\ldots,$$

and where $f(u)$ is a given function which may be written in the form

$$f(u)=c_1u^{a_1}+c_2u^{a_2}+c_3u^{a_3}+\ldots,\quad \alpha \lesseqgtr a_1<a_2<a_3<\ldots;$$

α and the a's being positive or negative, entire or fractional, subject to the given relations.

First consider the special case

$$c_iu^{a_i}=\int_0^u(\beta_0+\beta_1\frac{y}{u}+\beta_2\frac{y^2}{u^2}+\ldots)u^\alpha\,\phi_i(y)\,dy. \tag{2}$$

Now

$$\int_0^u(\beta_0+\beta_1\frac{y}{u}+\beta_2\frac{y^2}{u^2}+\ldots)\delta_iy^{a_i-\alpha-1}dy$$

$$=\delta_iu^\alpha\left(\frac{\beta_0}{a_i-\alpha}y^{a_i-\alpha}+\frac{\beta_1}{a_i-\alpha+1}\,\frac{y^{a_i-\alpha+1}}{u}+\frac{\beta_2}{a_i-\alpha+2}\,\frac{y^{a_i-\alpha+2}}{u^2}+\ldots\right)\Big]_0^2$$

$$=\delta_i\,s_i\,u^{a_i},\text{ where} \tag{3}$$

$$s_i=\frac{\beta_0}{a_i-\alpha}+\frac{\beta_1}{a_i-\alpha+1}+\frac{\beta_2}{a_i-\alpha+2}+\frac{\beta_3}{a_i-\alpha+3}+\ldots. \tag{4}$$

Equating the first member of (2) with the second of (3), we have

$$\delta_i=\frac{c_i}{s_i}.\quad \therefore \phi_i(y)=\frac{c_i}{s_i}y^{a_i-\alpha+1}.$$

Evidently, then, a solution of (1) is

$$\phi(y)=\sum_i\phi_i(y);\text{ or }\phi(y)=\sum_i\frac{c_i}{s_i}x^{a_i-\alpha-1},$$

where i runs over all the subscripts of c in the expression for $f(u)$ and s_i is defined by equation (4).

NOTE.—Volterra has studied a more general problem. See *Encyklop. d. Math. Wissensch.*, II, p, 808. The problem in the present form affords the solution of several problems in practical hydrodynamics.